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Locus of viewpoints from which a conic appears circular

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Abstract

We know that circular shapes we encounter in daily life may appear to be elliptical from some viewing points. It is reasonable to expect that an ellipse may appear to be circular from certain viewpoints. In this paper, we investigate the locus of viewpoints from which an ellipse appears circular. It will be shown that the locus of viewpoints is a hyperbola passing through the two foci of the ellipse. Conversely, the locus of viewpoints from which a hyperbola looks circular is an ellipse passing through the two foci of the hyperbola. Further, the locus of viewpoints from which a parabola looks circular is itself a parabola, passing through the focus of the original parabola. There is a simple duality between the object to be observed and the observer.

1 Introduction

Let us imagine observing a circle, drawn on the ground, from various viewpoints. The circle will look like an ellipse except when viewed from directly above. Next, let us imagine observing an ellipse drawn on the ground. The ellipse may look like a circle from certain viewpoints. This is the starting point of our investigation. Mathematically, a viewpoint from which an ellipse appears to be circular corresponds to the vertex of a right circular cone containing the ellipse. Hence, the set of viewpoints from which an ellipse looks circular is the set of vertices of a right circular cone containing the ellipse. This set of viewpoints results in a 3D space curve in the form of a hyperbola passing through the two foci of the ellipse, as shown in Figure 1.

Conversely, consider the case when we observe a hyperbola drawn on the ground. There are many viewpoints from which the two branches of the hyperbola will look like two circular arcs. A viewpoint from which a hyperbola appears as two circular arcs corresponds to the vertex of a right circular cone containing the hyperbola. (The two circular arcs are complements of each other, in the sense that we can join the two arcs to make a complete circle). In this case, the locus of viewpoints is also a 3D space curve, in the form of an ellipse passing through the two foci of the hyperbola, as shown in Figure 2.

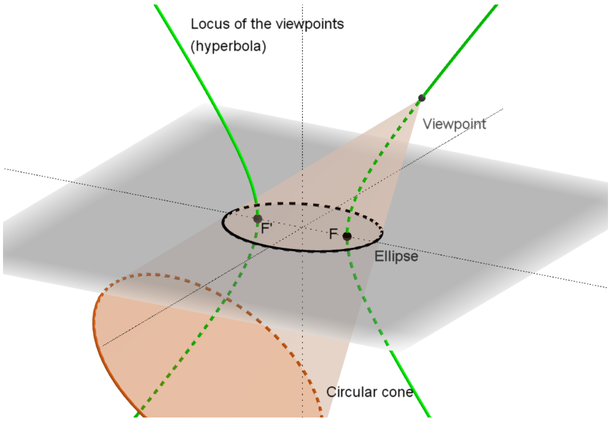

Figure 1: Ellipse on the ground.

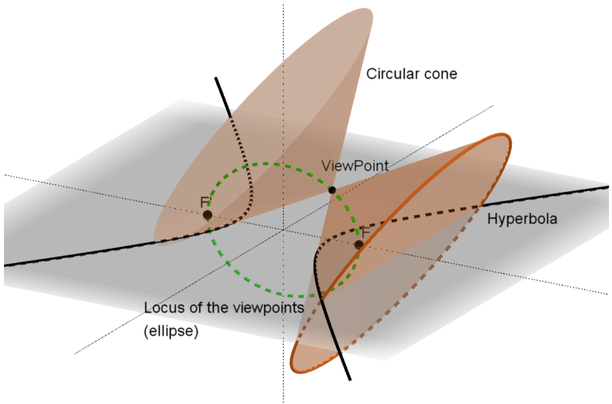

Figure 2: Hyperbola on the ground.

Here, we can note a certain symmetry or duality. Let E be an ellipse on the ground. The locus of viewpoints from which E looks circular is a hyperbola H. If, in turn, we look at the locus H from an arbitrary point on E, then H looks circular (more precisely, two complemental arcs). Conversely, let H' be a hyperbola on the ground. The locus of viewpoints from which H' looks circular is an ellipse E'. If, in turn, we look at the locus E' from an arbitrary point on H', then E' looks circular, as shown in Figure 6. Thus, there exists a simple duality between the object to be observed and the observer.

In the case of a parabola drawn on the ground, there are many viewpoints from which the parabola looks circular, constituting a tangent to the horizontal line at infinity as shown in Figures 7 and 10. The locus of viewpoints from which the parabola looks circular is itself a parabola with the same shape as the original parabola. In this sense, the parabola is "self-dual".

Our discussion is done in the three-dimensional Euclidean space $\mathbb{R}^3$ with Cartesian coordinates. Assume that the object to be observed ("on the ground") is in the XY-plane. To simplify our discussion, we also assume that viewpoint V is not in the XY-plane, that is, $V_z \neq 0$.

2 The locus of viewpoints

To investigate the locus of viewpoints, Dandelin's construction ([1] p.227, [2] pp.9, [3] pp.87–92) plays an important role. A conic C is obtained by intersecting a right circular cone K with a plane H. If C is an ellipse or a hyperbola, there are two spheres which are tangent to K and H, as shown in Figure 3. For each sphere, the point tangent to H is one of the foci of C. The vertex V of K lies on the line connecting the centers of the two spheres (the line is the axis of K).

Theorem 1 *Let C be the ellipse in the XY-plane defined by the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants such that $a > b$. Then, the locus of viewpoints from which C looks circular is the hyperbola in the XZ-plane determined by the equation

$$\frac{x^2}{c^2} - \frac{z^2}{b^2} = 1$$

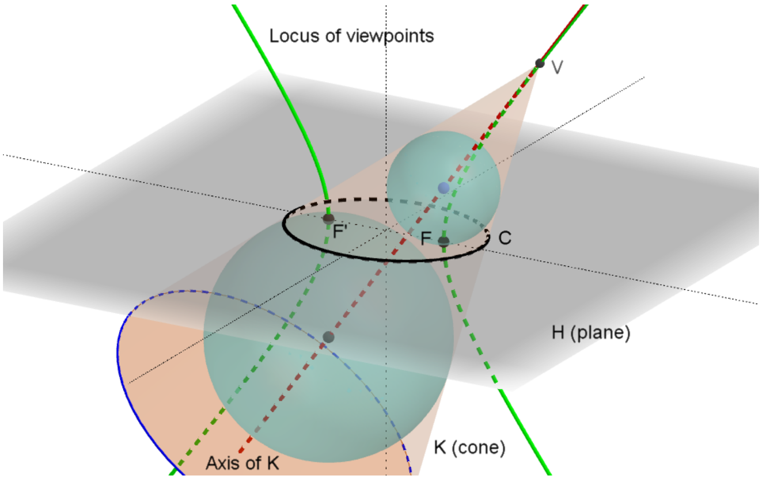

Figure 3: Dandelin's construction.

where $c = \sqrt{a^2 - b^2}$ *(Figure 1).*

Proof. The proof is done in two steps. First, we consider the necessary condition of the locus, after which we check that the condition is sufficient - that is, that the ellipse looks circular from any viewpoint on the locus. Suppose that V is a viewpoint from which the ellipse C looks circular. Then, there exists a right circular cone K with vertex V containing C. Let $A = (a, 0, 0)$ and $A' = (-a, 0, 0)$ be the two points on the major axis of C. Let $F = (c, 0, 0)$ and $F' = (-c, 0, 0)$ be the two foci of the ellipse where $c = \sqrt{a^2 - b^2}$. There are two spheres S and S' tangent to K and the XY-plane at F and F', respectively. Since the centers of both spheres are in the XZ-plane, V is also in the XZ-plane. Hence, the locus of viewpoints lies on the XZ-plane. Figure 4 shows the cross section in the XZ-plane. If the radius of S is smaller than that of S', the cross section of S is the inscribed circle

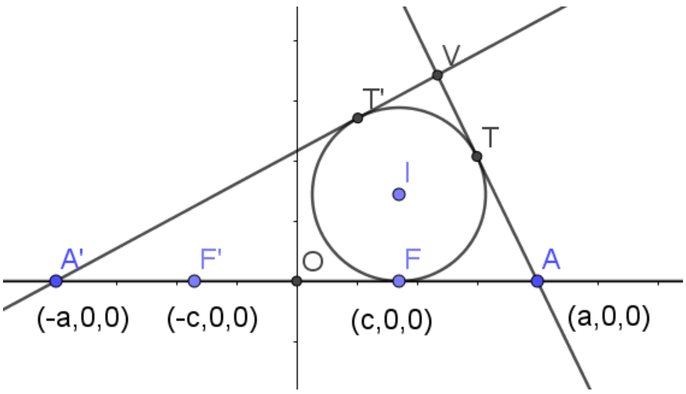

Figure 4: Cross section in the XZ-plane (elliptic case).

of $\triangle AVA'$ in the XZ-plane. Let T be the tangent point of S and generator VA, and T' be the tangent point of S and generator VA'. Then, $\overline{AT} = \overline{AF} = a - c$, $\overline{A'T'} = \overline{A'F} = a + c$, and $\overline{VT} = \overline{VT'}$.

Therefore,

$$|\overline{VA}-\overline{VA'}| = |\overline{AT}-\overline{A'T'}| = 2c.$$

Similarly, we can derive the equation $|\overline{VA}-\overline{VA'}| = 2c$ for the case in which the radius of S' is smaller than that of S. Hence, we demonstrate the necessary condition: V is on the hyperbola C' with the foci A and A', passing through F and F' in the XZ-plane. It is then easy to check that the equation of the hyperbola C' is $x^2/c^2 - z^2/b^2 = 1$.

Next, we prove that ellipse C looks circular from any point on the hyperbola C' except for F and F'. The ellipse C in the XY-plane is parametrized such that $P = (a\cos\theta, b\sin\theta, 0)$ where $\theta \in [0, 2\pi)$ is a parameter of the ellipse C. Let $V = (\pm c\cosh t, 0, b\sinh t)$ be a point on the hyperbola C' with non-zero parameter $t \in \mathbb{R}$. Let $\overrightarrow{u} = (\pm c\sinh t, 0, b\cosh t)$ be a tangent vector to the hyperbola C' at V. (In fact, the angle bisector of $\angle AVA'$ is parallel to $\overrightarrow{u}$). Let $\overrightarrow{v} = (a\cos\theta \mp c\cosh t, b\sin\theta, -b\sinh t)$ be the vector from V to P. Let φ be the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$. By direct calculation using $a^2 = b^2 + c^2$,

$$\begin{aligned}
|\overrightarrow{u}|^2 &= c^2\sinh^2 t + b^2\cosh^2 t,\\
|\overrightarrow{v}|^2 &= (a\cosh t \mp c\cos\theta)^2,\\
\overrightarrow{u}\cdot\overrightarrow{v} &= -a\sinh t(a\cosh t \mp c\cos\theta).
\end{aligned}$$

Since,

$$\cos^2\varphi = \frac{(\overrightarrow{u}\cdot\overrightarrow{v})^2}{|\overrightarrow{u}|^2|\overrightarrow{v}|^2} = \frac{a^2\tanh^2 t}{c^2\tanh^2 t + b^2} \in (0,1),$$

the angle φ does not depend on θ. Therefore, P is on the right circular cone with the vertex V, whose axis is the tangent line of the hyperbola C' at V, and whose vertex angle is φ. ■

The next theorem is the case of the hyperbola in the XY-plane. The proof is similar to that of Theorem 1.

Theorem 2 *Let C be the hyperbola in the XY-plane defined by the equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where a and b are positive constants. Then, the locus of viewpoints from which C looks circular is the ellipse in the XZ-plane determined by the equation

$$\frac{x^2}{c^2} + \frac{z^2}{b^2} = 1$$

where $c = \sqrt{a^2+b^2}$ (Figure 2).

Proof. First, we consider the necessary condition of the locus. Suppose that V is a viewpoint from which the hyperbola C looks circular. Then, there exists a right circular cone K with vertex V containing C. Let $A = (a, 0, 0)$ and $A' = (-a, 0, 0)$ be the two points on the major axis of C. Let $F = (c, 0, 0)$ and $F' = (-c, 0, 0)$ be the two foci of the hyperbola where $c = \sqrt{a^2+b^2}$. There are two spheres S and S' tangent to K and the XY-plane at F and F', respectively. Since the centers of

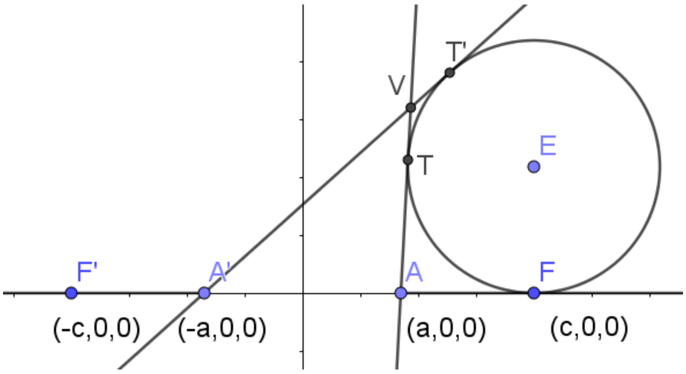

Figure 5: Cross section in the XZ-plane (hyperbolic case).

both spheres are in the XZ-plane, V is also in the XZ-plane. Hence, the locus of viewpoints lies on the XZ-plane. Figure 5 shows the cross section in the XZ-plane. In the XZ-plane, the cross section of S is one of the escribed circles of $\triangle AVA'$. Let T be the tangent point of S and generator VA, and T' be the tangent point of S and generator VA'. Then, $\overline{AT} = \overline{AF} = c - a$, $\overline{A'T'} = \overline{A'F} = a + c$, and $\overline{VT} = \overline{VT'}$. Therefore,

$$\overline{VA} + \overline{VA'} = \overline{AT} + \overline{A'T'} = 2c.$$

Hence, we demonstrate the necessary condition: V is on the ellipse C' with the foci A and A', passing through F and F' in the XZ-plane. It can be shown that the equation of the ellipse C' is $x^2/c^2 + z^2/b^2 = 1$.

Next, we prove that the hyperbola C looks circular from any point on the ellipse C' except for F and F'. The hyperbola C in the XY-plane is parametrized such that $P = (\pm a\cosh s, b\sinh s, 0)$ where $s \in \mathbb{R}$ is a parameter of the hyperbola C. Let $V = (c\cos\theta, 0, b\sin\theta)$ be a point on the ellipse C' with parameter $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Let $\overrightarrow{u} = (-c\sin\theta, 0, b\cos\theta)$ be a tangent vector to the ellipse C' at V. (In fact, the angle bisector of $\angle AVA'$ is perpendicular to $\overrightarrow{u}$). Let $\overrightarrow{v} = (\pm a\cosh s - c\cos\theta, b\sinh s, -b\sin\theta)$ be the vector from V to P. Let φ be the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$. By direct calculation using $c^2 = a^2 + b^2$,

$$\begin{aligned} |\overrightarrow{u}|^2 &= c^2\sin^2\theta + b^2\cos^2\theta, \\ |\overrightarrow{v}|^2 &= (a\cos\theta \mp c\cosh s)^2, \\ \overrightarrow{u}\cdot\overrightarrow{v} &= a\sin\theta(a\cos\theta \mp c\cosh s). \end{aligned}$$

Since,

$$\cos^2\varphi = \frac{(\overrightarrow{u}\cdot\overrightarrow{v})^2}{|\overrightarrow{u}|^2|\overrightarrow{v}|^2} = \frac{a^2\tan^2\theta}{c^2\tan^2\theta + b^2} \in (0,1),$$

the angle φ does not depend on s. Therefore, P is on the right circular cone with the vertex V, whose axis is the tangent line of the ellipse C' at V, and whose vertex angle is φ. ■

Theorems 1 and 2 imply a duality between the object to be observed and the observer. Figure 6 shows a dual relation between an ellipse E in the XZ-plane and a hyperbola H in the XY-plane. The

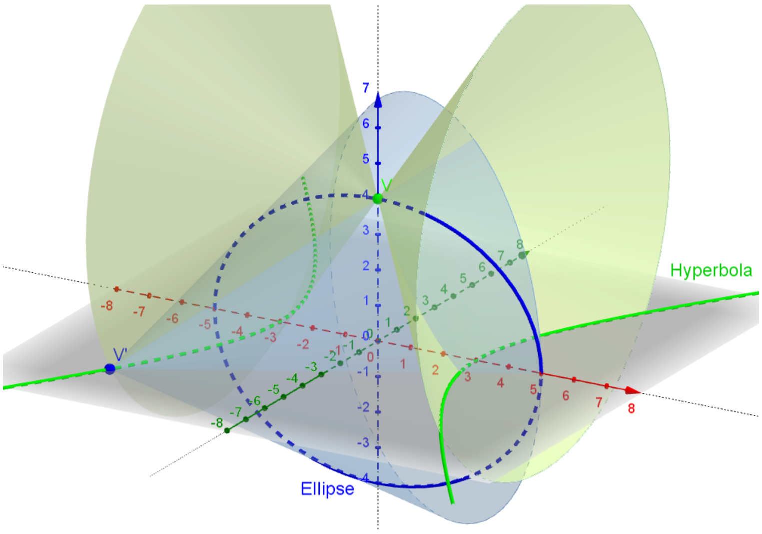

Figure 6: Duality between ellipse and hyperbola.

equation of the ellipse E is $x^2/5^2 + z^2/4^2 = 1$ in the XZ-plane, and the equation of the hyperbola H is $x^2/3^2 - y^2/4^2 = 1$ in the XY-plane. E looks circular from any point on H, and H looks circular from any point on E.

Finally, we show that the prediction that the locus of viewpoints from which a parabola looks circular is also a parabola, is correct. Note that for a parabola C, we can set up a system of Cartesian coordinates on the Euclidean plane as follows: the origin is at the midpoint of the vertex of C and the focus of C, and the X-axis is the axis of symmetry of C. Then, the equation of C is $x = \dfrac{y^2}{8c} - c$ for a real non-zero number c. $A = (-c, 0)$ is the vertex of C, and $F = (c, 0)$ is the focus of C. Then, the equation of the directrix of C is $x = -3c$.

Theorem 3 *Let C be the parabola in the XY-plane defined by the equation*

$$x = \frac{y^2}{8c} - c,$$

where c is a non-zero constant. Then, the locus of viewpoints from which C looks circular is the parabola determined by the equation (Figure 7)

$$x = -\frac{z^2}{8c} + c.$$

Proof. First, we consider the necessary condition of the locus. Suppose that V is a viewpoint from which the parabola C looks circular. Then, there exists a right circular cone K with vertex V containing C. Let $A = (-c, 0, 0)$ be the vertex of C, and $F = (c, 0, 0)$ the focus of C. There exists only one sphere S tangent to K and the XY-plane at F. Notice that there is one generator ℓ_∞ of the cone K

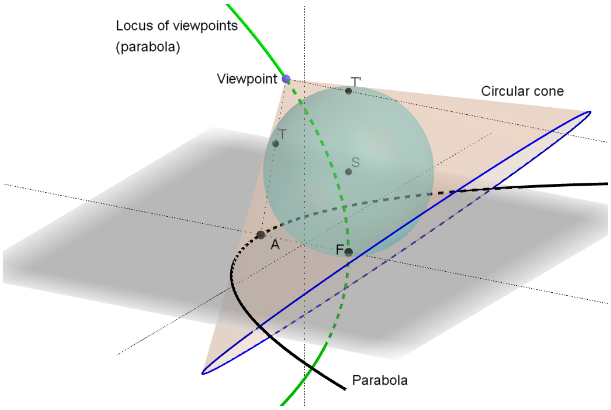

Figure 7: Parabola on the ground.

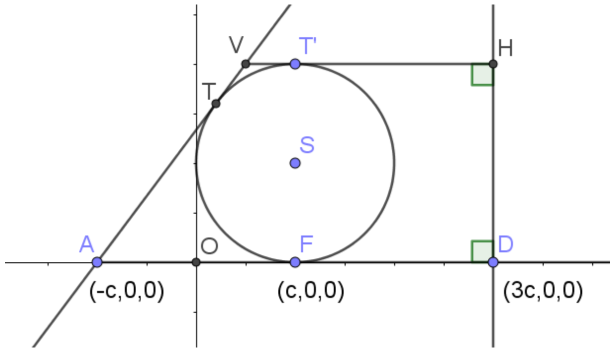

Figure 8: Cross section in the XZ-plane (parabolic case).

such that ℓ_∞ is parallel to the XY-plane. Let us show that ℓ_∞ is also parallel to AF. Let P be a point on C. When P goes to infinity, then,

$$\begin{aligned}\lim_{P\to\infty} AP &= AF,\\ \lim_{P\to\infty} VP &= \ell_\infty,\end{aligned}$$

and hence, the two lines AF and ℓ_∞ have the same direction, so ℓ_∞ is parallel to AF. Let T' be the tangent point of ℓ_∞ to S. Then T' is the reflection of F with respect to the center of S, otherwise $\ell_\infty(= VT')$ is not parallel to the XY-plane. Therefore, T' is in the XZ-plane, and V is also in the XZ-plane because $VT' = \ell_\infty // AF$. Consequently, the locus of viewpoints lies on the XZ-plane. Figure 8 shows the cross section in the XZ-plane.

Let T be the tangent point of S and generator VA. Let d be the line passing through $D = (3c, 0, 0)$ parallel to the Z-axis. Then, $\overline{AT} = \overline{AF} = 2c$, $\overline{T'd} = \overline{FD} = 2c$, and $\overline{VT} = \overline{VT'}$. Therefore,

$$\overline{VA} = \overline{VT} + \overline{AT} = \overline{VT'} + 2c = \overline{VT'} + \overline{T'd} = \overline{Vd}.$$

Hence, we demonstrate the necessary condition: V is on the parabola C' with the focus A, passing through F in the XZ-plane (d is the directrix of the parabola C'). It can be shown that the equation of the parabola C' is $x = -\dfrac{z^2}{8c} + c$.

Next, we prove that the parabola C in the XY-plane looks circular from any point on the parabola C' except for F. The parabola C in the XY-plane is parametrized such that $P = \left(\dfrac{s^2}{8c} - c, s, 0\right)$ where $s \in \mathbb{R}$ is a parameter of the parabola C. Let $V = \left(-\dfrac{t^2}{8c} + c, 0, t\right)$ be a point on the parabola C' with non-zero parameter $t \in \mathbb{R}^\times (= \mathbb{R} - \{0\})$. Let $\overrightarrow{u} = (t, 0, -4c)$ be a tangent vector to the parabola C' at V. (In fact, the angle bisector of $\angle AVT'$ is parallel to $\overrightarrow{u}$). Let $\overrightarrow{v} = \left(\dfrac{s^2 + t^2}{8c} - 2c, s, -t\right)$ be the vector from V to P.

Let φ be the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$. By direct calculation,

$$\begin{aligned} |\overrightarrow{u}|^2 &= t^2 + 16c^2, \\ |\overrightarrow{v}|^2 &= \left(\frac{s^2+t^2}{8c} + 2c\right)^2, \\ \overrightarrow{u} \cdot \overrightarrow{v} &= t\left(\frac{s^2+t^2}{8c} + 2c\right). \end{aligned}$$

Since,

$$\cos^2\varphi = \frac{(\overrightarrow{u} \cdot \overrightarrow{v})^2}{|\overrightarrow{u}|^2|\overrightarrow{v}|^2} = \frac{t^2}{t^2+16c^2} \in (0,1),$$

the angle φ does not depend on s. Therefore, P is on the right circular cone with the vertex V, whose axis is the tangent line of the parabola C' at V, and whose vertex angle is φ. ■

Remark 4 *If a parabola is defined by a standard form $y = ax^2$ with a constant a, then the locus is given by the equation $y = -az^2 + \frac{1}{4a}$. Note that the locus passes through the focus $\left(0, \frac{1}{4a}\right)$ of the original parabola in the XY-plane.*

It is straightforward to show the duality in the case of parabola. Let C be a parabola such that its focus is at F and its vertex is at F^*. Let H be the plane on which C lies, and H^* be the plane perpendicular to H including the line FF^*. Let C^* be the locus of viewpoints from which C looks circular. Then, C^* lies on H^* with its focus at F^* and its vertex at F. Let C^{**} be the locus of viewpoints from which C^* looks circular. Then, C^{**} lies on H with its focus at F and its vertex at F^*, that is, C^{**} is in fact C.

3 Closing remarks

In this paper, we have explored the locus of viewpoints from which a conic looks circular. There is a simple duality between the conic and the locus.
If you were to traverse the locus of viewpoints using a drone with a camera, as shown in Figure 9, the drone would move toward the virtual center (the axis of the cone) of the virtual circle (formed by the base of the cone), since the axis of the cone is the tangent of the locus.
We cannot accurately represent the virtual circle on the flat surface of the page, as our virtual circle and its virtual center exist on a spherical section (curved screen) centered at the viewpoint. Hence, Figure 10 is not an accurate drawing, merely a rough image of the scene.

4 Acknowledgement

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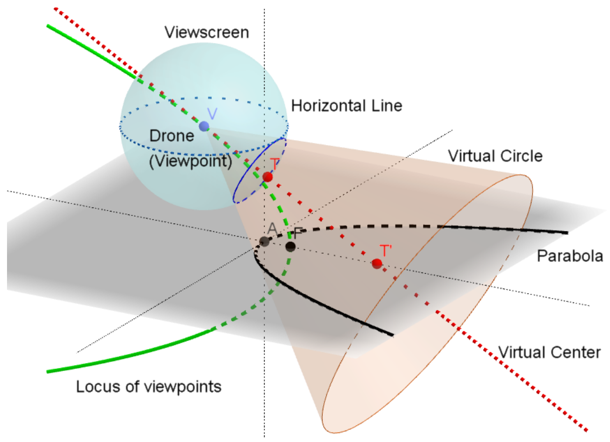

Figure 9: Viewscreen.

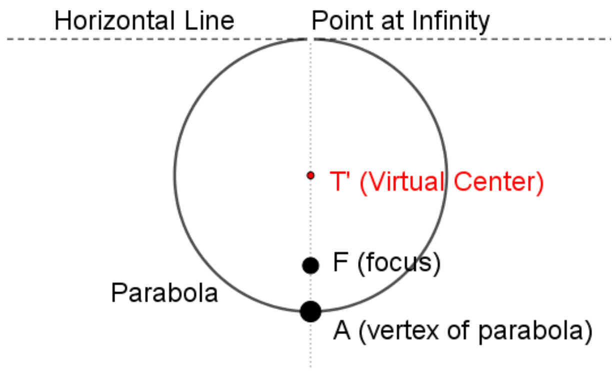

Figure 10: Image of the scene.

5 Supplementary Electronic Materials

[S1] Figure 1 in Section 2.
[S2] Figure 2 in Section 2.
[S3] Figure 3 in Section 2.
[S4] Figure 6 in Section 2.
[S5] Figure 7 in Section 2.
[S6] Figure 9 in Section 3.

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